Further aspects of regular spatial hexagons

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In the following, we refer to article ≪Regular spatial hexagons≫ (see also *Elem. Math.*) with regard to concepts, notations, references to theorems etc. In particular, we consider hexagons with consecutive vertices v_1, v_2, …, v_6, six diagonals q between a vertex and the next but one, and three remaining diagonals $x = \overline{v_1 v_4}$, $y = \overline{v_2 v_5}$, and $z = \overline{v_3 v_6}$.

1. Pentas

Consider pentas with $z = 0$.

a. Show that the diagonals x and y are given as follows:

$$0 < x < \sqrt{3}, \;\; y = 2\sqrt{\frac{x^2-3}{x^2-4}}.$$

b. What about $x < y$, $x = y$, $x > y$, and the boundary cases $x = 0$ and $x = \sqrt{3}$?

c. There are two pentas with the same vertices. In which way are they related?

d. Examine the convex hull of pentas including the special cases.

e. Which penta has the largest volume of the convex hull?

2. Relations between diagonals of flexible hexagons

a. Verify the symmetric relation

$$\begin{aligned} 0 \;=\; &+ x^4y^4 \\ &+ 4(q^2+1)x^3y^3 - 2(q^2+1)x^2y^2(x+y)^2 \\ &+ 6(q^4+1)x^2y^2 - 4(q^2-1)^2xy(x+y)^2 + (q^2-1)^2(x+y)^4 \\ &- 4(q^2-1)^3xy + 2(q^2-1)^3(x+y)^2 \\ &- (q^2-1)^2(q^2+1)(7q^2-1). \end{aligned}$$

Of course, there are corresponding relations with the diagonals y, z and x, z.

b. Determine once more the diagonals of boats, crosses, twist, and pentas.

3. Line segments on the prime symmetry axis of flexible hexagons

Let s_{xy} be the line segment between the midpoints of the diagonals x and y, and correspondingly s_{yz} and s_{xz} (all lying on the prime symmetry axis).

a. Show that

$$s_{xy} = \frac{1}{2}\sqrt{\frac{(x^2-q^2-1)(y^2-q^2-1)}{z^2-q^2-1}}.$$

b. Discuss the case $s_{xy} = s_{xz}$.

c. Show that $s_{xy} = s_{xz} + s_{yz}$ if and only if z is largest.

4. A Duality of flexible hexagons

Consider the representation of flexible hexagons by diagonal points, as shown in Figure 12. A straight line through the origin pierces the corresponding area in exactly two points, which represent what we call *dual* hexagons.

a. Determine for a boat with diagonals q, x, y, and z the associated diagonals q', x', y', and z' of the dual hexagon.

b. Determine q' for crosses and twists.

5. Vertex coordinates of flexible *s*-hexagons based on the prime symmetry

We refer to Property 8. Let $v_1 = (1, 0, 0)$ and $v_2 = (a, b, h_y)$ with $h_y \geq 0$ be two vertices of flexible s-hexagons where the prime symmetry axis coincides with the third coordinate axis. Furthermore, assume that

$$-1 < a < \frac{1}{2} \;\; (a \neq 0) \;\; \text{and} \;\; \sqrt{|a|\,(a+1)} \leq b \leq \sqrt{1-a^2}.$$

a. Determine the vertices $v_1, v_2, \ldots, v_6$ of these flexible s-hexagons and show that their diagonals satisfy $x \geq y \geq z$.

b. What is the significance of the boundary values of b?

6. Largest convex hull of hexagons (with side length 1)

a. Determine the largest volume of the convex hull for crowns, stars, boats, crosses, and twists.

b. What can be said about all hexagons?

Solutions

1. a. Use the Pythagorean theorem or insert $q = 1$ in Theorem 14.
Note that x and y are exchangeable, i.e., we have a self-inverse function.

b. $x < y$ if $0 < x < \sqrt{2}$, $\quad x = y$ if $x = \sqrt{2}$, $\quad x > y$ if $\sqrt{2} < x < \sqrt{3}$.
Boundary cases: rhombus with two double vertices for both $x = 0$ and $x = \sqrt{3}$.

c. The two chiral pentas are enantiomeric (mirrored to each other). One of them is obtained from the other by a reflection across the plane that is spanned by the prime symmetry axis and one of the perpendicular diagonals x or y.

d. In general, the convex hull is a hexahedron (polyhedron with six faces), and it has the symmetry group $\mathrm{C_{2v}}$ (Schoenflies symbol). Four faces are equilateral triangles having the double vertex as a common point (and forming a sort of cupola, as shown in Figure E21), and two faces are isosceles triangles. Special cases: $x = 1$ gives a deltahedron (polyhedron whose faces are congruent equilateral triangles) with symmetry group $\mathrm{D_{3h}}$, and $x = \sqrt{2}$ gives half of an octahedron with symmetry group $\mathrm{C_{4v}}$.

e. One obtains congruent pentas with $x = \sqrt{\frac{3}{2}}$ or $x = 2\sqrt{\frac{3}{5}}$, and volume $\frac{1}{4}$.

2. a. Square twice both sides of the corresponding equation in Theorem 14.

b. These special cases are obtained by equating two of the diagonals x, y, and z, and by setting $q = 1$, respectively.

3. a. Needs an extensive calculation by using the coordinates of Theorem 13. Note that the formula is also true in the case of the limit $z \to \sqrt{1+q^2}$ where the denominator becomes 0.

b. One obtains with $s_{yz} \neq 0$ crosses or twists, and with $s_{yz} = 0$ boats. In all cases, a symmetry maps s_{xy} onto s_{xz}. Furthermore, it holds $s_{xy}\, s_{xz} = \frac{1}{4}(x^2 - q^2 - 1)$, which implies that y and z intersect each other only in the case of a boat.

c. Consider in all flexible hexagons with a given q the longest diagonal z. It lies in the symmetry plane of a cross or on a symmetry axis of a twist, both impliying $s_{xy} = s_{xz} + s_{yz}$. A continuous transformation of the considered hexagons preserves this equality until $x = y$ or $x = z$, which applies in a boat.

4. a. $q' = \sqrt{\dfrac{3 - q^2}{1 + q^2}}$ (self-inverse function), $\; d' = \dfrac{2d}{1 + q^2}$ where d is x, y, or z (implying $dd' = 2$ for the longer diagonals d). The dual hexagon of a boat is a boat as well.

b.

$$q' = \sqrt{\frac{3 + 5q^2 + (1 + q^2)\sqrt{3\,(3 + 4q^2 - 4q^4)}}{2\,(1 + 3q^2 + 3q^4)}} \quad \text{for } q < 1,$$

$$q' = \sqrt{\frac{q\,\big(5q - 3q^3 + (1 + q^2)\sqrt{3\,(4 - q^2)}\big)}{2\,(1 + 3q^4)}} \quad \text{for } q > 1.$$

The union of these two functions is self-inverse. The dual hexagon of a cross is a twist, and conversely.

5. a. Calculation leads to

$$v_{1,4} = (\pm 1, 0, 0),\ \ v_{2,5} = (\pm a, \pm b, h_y),\ \ v_{3,6} = (\mp a, \pm c, -h_z),\ \text{where}$$

$$c = \frac{a\,(a{+}1)}{b}\ \text{(follows immediately from the p-hexagon property),}$$

$$h_y = \frac{1}{\sqrt{3}\,b}\sqrt{2a^2(a{+}1)^2 + (a^2{-}1)b^2 - b^4 + 2r},$$

$$h_z = \frac{1}{\sqrt{3}\,b}\sqrt{-a^2(a{+}1)^2 + (a^2{-}1)b^2 + 2b^4 + 2r},\ \text{with}$$

$$r = \sqrt{a^4(a{+}1)^4{+}a^2(a{-}1)(a{+}1)^3b^2{+}(1{-}2a)(a{+}1)^2b^4{+}(a^2{-}1)b^6{+}b^8}.$$

By taking into account the ranges of the parameters a and b, it is shown that the diagonals $x{=}2$, $y{=}2\sqrt{a^2{+}b^2}$, and $z{=}2\sqrt{a^2{+}c^2}$ satisfy $x \geq y \geq z$. The side s and diagonal q of the resulting s-hexagons are given as follows:

$$s = \frac{\sqrt{2}}{\sqrt{3}\,b}\sqrt{a^2(a{+}1)^2 + (2a{-}1)(a{-}1)b^2 + b^4 + r},$$

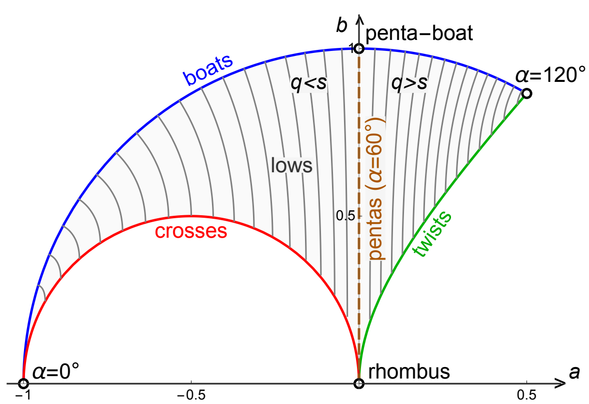
$$q = \frac{1}{\sqrt{3}\,b}\sqrt{2a^2(a{+}1)^2 + 3(a{+}1)^2b^2 + (a^2{-}1)b^2 + 2b^4 + 2r}.$$

These s-hexagons represent all classes of similar flexible s-hexagons.

b. The lower bound $b{=}\sqrt{|a|\,(a+1)}$ leads to crosses for $a{<}0$ and to twists for $a{>}0$, the upper bound $b{=}\sqrt{1-a^2}$ to boats.

c. For $a{=}0$ and $b{\neq}0$, one obtains pentas (for $b{=}1$ a penta-boat). The value $a{=}{-}1$ gives a line segment with two triple vertices and $a{=}\frac{1}{2}$ a planar (regular) hexagon. The case $(a,b){=}(0,0)$ needs some limit considerations in order to obtain a rhombus with two double vertices: $v_{1,4} = (\pm 1, 0, 0)$, $v_{2,5} = (0, 0, 1/\sqrt{3})$, $v_{3,6} = (0, 0, -1/\sqrt{3})$.

The following figure illustrates the results; each black line in the area represents hexagons with a fixed $\frac{q}{s}$ (iso-q/s-line):



6. a. crown: $q = \sqrt{2}$, volume $V = \frac{2}{3} \approx \mathbf{0.6667}$;

star: $q = \sqrt{\frac{2}{3}}$, $V = \frac{1}{6} \approx \mathbf{0.1667}$;

boat: $q = \sqrt{\frac{1}{10}(9+\sqrt{201})}$, $V = \frac{1}{50}\sqrt{\frac{1}{30}(10563+737\sqrt{201})} \approx \mathbf{0.5293}$;

cross: $q = \sqrt{\frac{2}{3}}$, $V = \frac{2}{9} \approx \mathbf{0.2222}$;

twist: $q \approx 1.5059$ (solution of the equation $3q^8 - 21q^6 + 42q^4 - 24q^2 + 4 = 0$),
$V \approx \mathbf{0.6481}$.

b. Among all flexible hexagons, we could not find an exact result, but based on numerical computation, we presume that the largest V appears in the case of the twist. This would imply that, among all hexagons, the maximum volume will be assumed in the crown with $V = \frac{2}{3}$.

The final figure shows the volume V as a function of the diagonals q and x for flexible hexagons (with $y \geq x \geq z$ or $z \geq x \geq y$) as well as for rigid hexagons. There are two families of black lines, namely iso-q-lines and iso-V-lines. (For the largest V of pentas, see solution **1e**.)