

Further aspects of regular spatial heptagons

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In the following, we refer to the manuscript «Regular spatial heptagons based on symmetry» (see also [Elem. Math.](#)) with regard to concepts, notations, references to theorems etc. In particular, we consider heptagons with consecutive vertices $v_1, v_2, \dots, v_7$, side length 1, and seven diagonals of length q between a vertex and the next but one. The seven remaining diagonals are said to be the main diagonals.

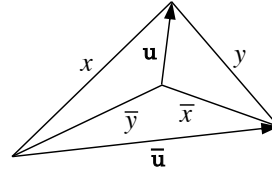
1. Special scalar products in heptagons

Let $\mathbf{u}_{i,j}$ be the vector that points in a heptagon from v_i to v_j (indices larger than 7 are understood modulo 7). Then, it holds the following:

$$\begin{aligned}\mathbf{u}_{k,k+3} \cdot \mathbf{u}_{k+1,k+2} &= 2 \mathbf{u}_{k,k+3} \cdot \mathbf{u}_{k+1,k+5} = q^2 - 1, \\ (\mathbf{u}_{k,k+3} + \mathbf{u}_{k,k+4}) \cdot \mathbf{u}_{k+2,k+5} &= 0.\end{aligned}$$

The equations are the consequence of a general property of a (possibly degenerate) tetrahedron according to the figure (e.g. see [1]):

$$\mathbf{u} \cdot \bar{\mathbf{u}} = \frac{1}{2} (x^2 + \bar{x}^2 - y^2 - \bar{y}^2).$$



2. Heptagons with $q = 1$ as limit figures

We consider the connected components for a fixed $q \neq 1$ according to Connectedness 4. With $q \rightarrow 1$, they become subsets of all heptagons with $q = 1$. Which heptagons of Theorem 4 represent these subsets? To answer this question, we use a notation based on the fact that there is a one-to-one correspondence between the incongruent plane-symmetric heptagons and the connected components.

Definition. $\mathcal{C}_{\text{large,up}}(q)$ is the connected component (of the corresponding q) containing the large and upper plane-symmetric heptagon; and accordingly for the other components.

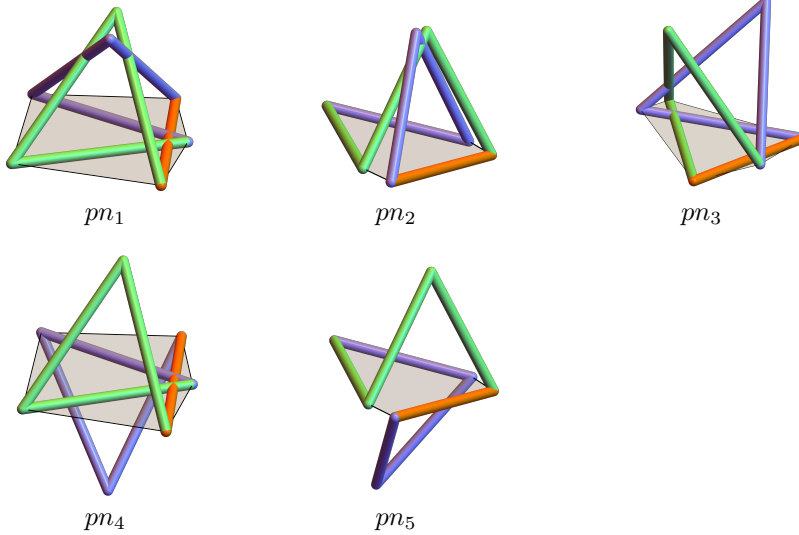
Taking into account Figures 3 and 4 as well as the remarks to Theorem 4, one obtains the following heptagons from Theorem 4:

- from $\lim_{q \uparrow 1} \mathcal{C}_{\text{large,up}}(q)$, those with v'_7 for $\varphi \in [-2\varphi_0, 0]$ and v''_7 for $\varphi \in [0, \pi - \varphi_0]$,
- from $\lim_{q \uparrow 1} \mathcal{C}_{\text{large,low}}(q)$, those with v'_7 for $\varphi \in [0, 2\varphi_0]$ and v''_7 for $\varphi \in [-\varphi_0, 0]$,
- from $\lim_{q \downarrow 1} \mathcal{C}_{\text{large,up}}(q)$, those with v'_7 for $\varphi \in [-2\varphi_0, 0]$ and v''_7 for $\varphi \in [-\varphi_0, 0]$,
- from $\lim_{q \downarrow 1} \mathcal{C}_{\text{large,low}}(q)$, those with v'_7 for $\varphi \in [\pi - 4\varphi_0, 2\varphi_0]$,
- from $\lim_{q \downarrow 1} \mathcal{C}_{\text{small,up}}(q)$, those with v'_7 for $\varphi \in [0, \pi - 4\varphi_0]$ and v''_7 for $\varphi \in [0, \pi - 2\varphi_0]$,
- from $\lim_{q \downarrow 1} \mathcal{C}_{\text{small,low}}(q)$, those with v''_7 for $\varphi \in [\pi - 2\varphi_0, \pi - \varphi_0]$.

3. Some special heptagons

We present three statements that can be verified by computation.

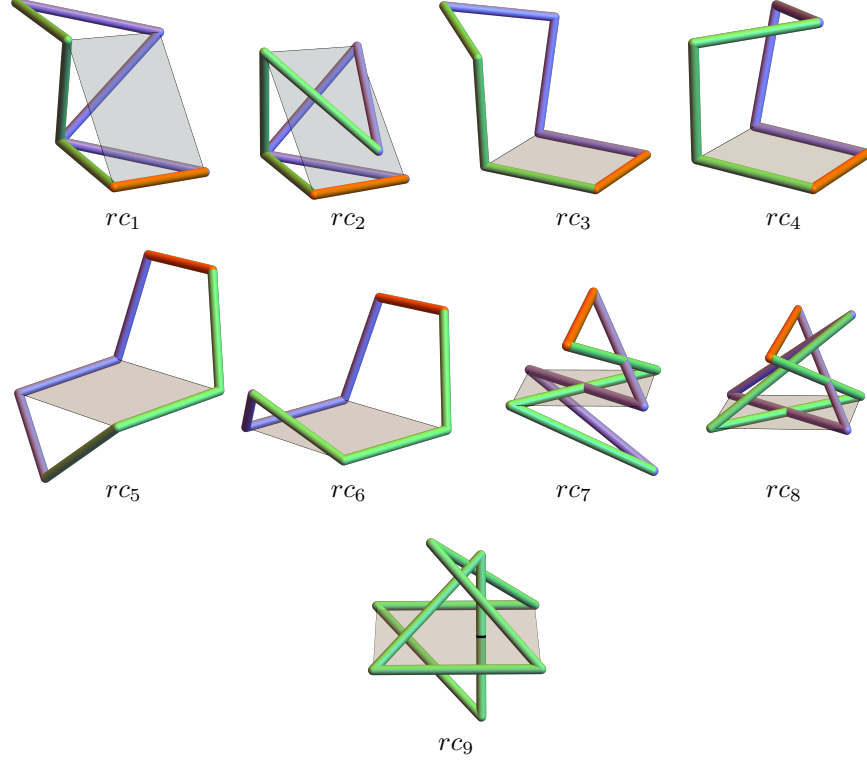
a. *There are only the following five incongruent heptagons with exactly five vertices in a plane.*



From the heptagons pn_i (penta), pn_1 , pn_2 , pn_3 are plane- and pn_4 , pn_5 line-symmetric. Further, pn_1 and pn_4 have the same angle $\alpha \approx 48.011^\circ$, and one heptagon is obtained from the other by reflecting a vertex and its adjacent sides on the pentagon plane. The same is true for pn_2 and pn_5 with the common angle $\alpha \approx 63.317^\circ$. The angle of pn_3 is $\alpha \approx 51.229^\circ$. All these symmetric heptagons are large and lower, except pn_2 which is small and upper.

We add that only pl_4 and lk have exactly six vertices in a plane (double vertex counted twice); and of course, the stars have seven of them.

b. *There are only the following nine incongruent heptagons with exactly four vertices forming a rectangle.*



The heptagons rc_i (rectangle) are plane-symmetric and appear always pairwise in the upper and lower form; an exception is only the asymmetric rc_9 .

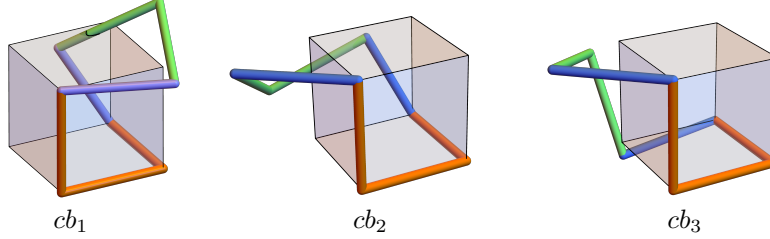
In the first row, we have rc_1 and rc_2 with angle $\alpha = 60^\circ$ (they are the same as pl_3 and pl_5 in [2]), and rc_3 and rc_4 with $\alpha = 90^\circ$.

The heptagons of the second row can be obtained without calculation as follows: For rc_5 and rc_6 , consider a regular pentagon and divide it along a diagonal q into two parts. Then, insert two new sides of length 1 between the parts in such a way that they are parallel. The heptagons rc_5 and rc_6 are now achieved by rotating both parts around q , where their common angle is $\alpha = 108^\circ$ ($q = \Phi$). The heptagons rc_7 and rc_8 are given by $\alpha = 36^\circ$ ($q = 1/\Phi$), and they can be obtained in a similar way, by starting with a pentagram and inserting new sides such that they are crossed. The four heptagons may be called *golden* heptagons.

Finally, the angle of the asymmetric rc_9 is $\alpha \approx 49.354^\circ$. Note that one side pierces the rectangle. Regarding the two possible connected components, we have:

$$rc_9 \in \mathcal{C}_{\text{large,up}}(q) \text{ with } q \approx 0.835.$$

c. There are only the following three incongruent heptagons with three consecutive sides as edges that determine a cube.



The asymmetric heptagons cb_i (cube) can be characterized with respect to the cube as follows: In cb_1 and cb_2 a fourth side lies on the surface but only in cb_1 a fifth side partially inside the cube, and in cb_3 no fourth side lies on the surface.

These heptagons belong to the two connected components with angle $\alpha = 90^\circ$ as follows:

$$cb_1 \in \mathcal{C}_{\text{large,low}}(q) \text{ and } cb_2, cb_3 \in \mathcal{C}_{\text{large,up}}(q) \text{ with } q = \sqrt{2}.$$

4. z-heptagons

By definition, a *z-heptagon* contains a z-shape, i.e., three consecutive sides in a plane as being two opposite sides and a diagonal of a parallelogram. Without loss of generality, we can assume that the z-shape is given by the vertex sequence $v_1 v_2 v_3 v_4$. And to avoid congruent z-heptagons with different diagonal lengths $\overline{v_2 v_6}$ (related to each other by the vertex permutation $(v_1 v_4)(v_2 v_3)(v_5 v_7)(v_6)$), we additionally require that $\overline{v_2 v_6} \leq \overline{v_3 v_6}$. Then, Figure 1 shows the curve resulting from the diagonal pairs $(q, \overline{v_2 v_6})$ of all z-heptagons. Each point of this curve represents exactly one z-heptagon, except the point at $q = 1$ that shows two of them. It follows that z-heptagons only exist for

$$q \in [\tilde{q}_1, \tilde{q}_3] \text{ with } \tilde{q}_1 \approx 0.7690 \ (\tilde{\alpha}_1 \approx 45.22^\circ) \text{ and } \tilde{q}_3 \approx 1.195 \ (\tilde{\alpha}_3 \approx 73.41^\circ).$$

Furthermore, for a fixed q , the number of incongruent z-heptagons is given as follows: one for $q \in [\tilde{q}_1, 1[\cup \{\tilde{q}_3\}$, two for $q \in \{1\} \cup]\tilde{q}_2, \tilde{q}_3[$, and three for $q \in]1, \tilde{q}_2]$, where $\tilde{q}_2 \approx 1.062 \ (\tilde{\alpha}_3 \approx 64.14^\circ)$; see the corresponding red colored boundaries in Figure 1.

In Figure 1, we also show a series of z-heptagons zh_i (each with drawn diagonal $v_2 v_6$). Note that $zh_3 = pl_3$ and $zh_4 = lk$ (see [2]). Further, there are exactly two incongruent z-heptagons with two double z-shapes, namely zh_3 and zh_{10} .

It should be mentioned that we only know the exact solutions in a few single cases: $\tilde{q}_1$ and $\tilde{q}_2$ are the absolute values of the real solutions of

$$11x^6 + 28x^5 - 5x^4 - 32x^3 + 24x^2 + 16x - 20 = 0,$$

and zh_{10} is determined by $\hat{q} = \sqrt{\frac{1}{2} + \frac{7}{2\sqrt{17}}} \approx 1.161 \ (\alpha \approx 71.00^\circ)$.

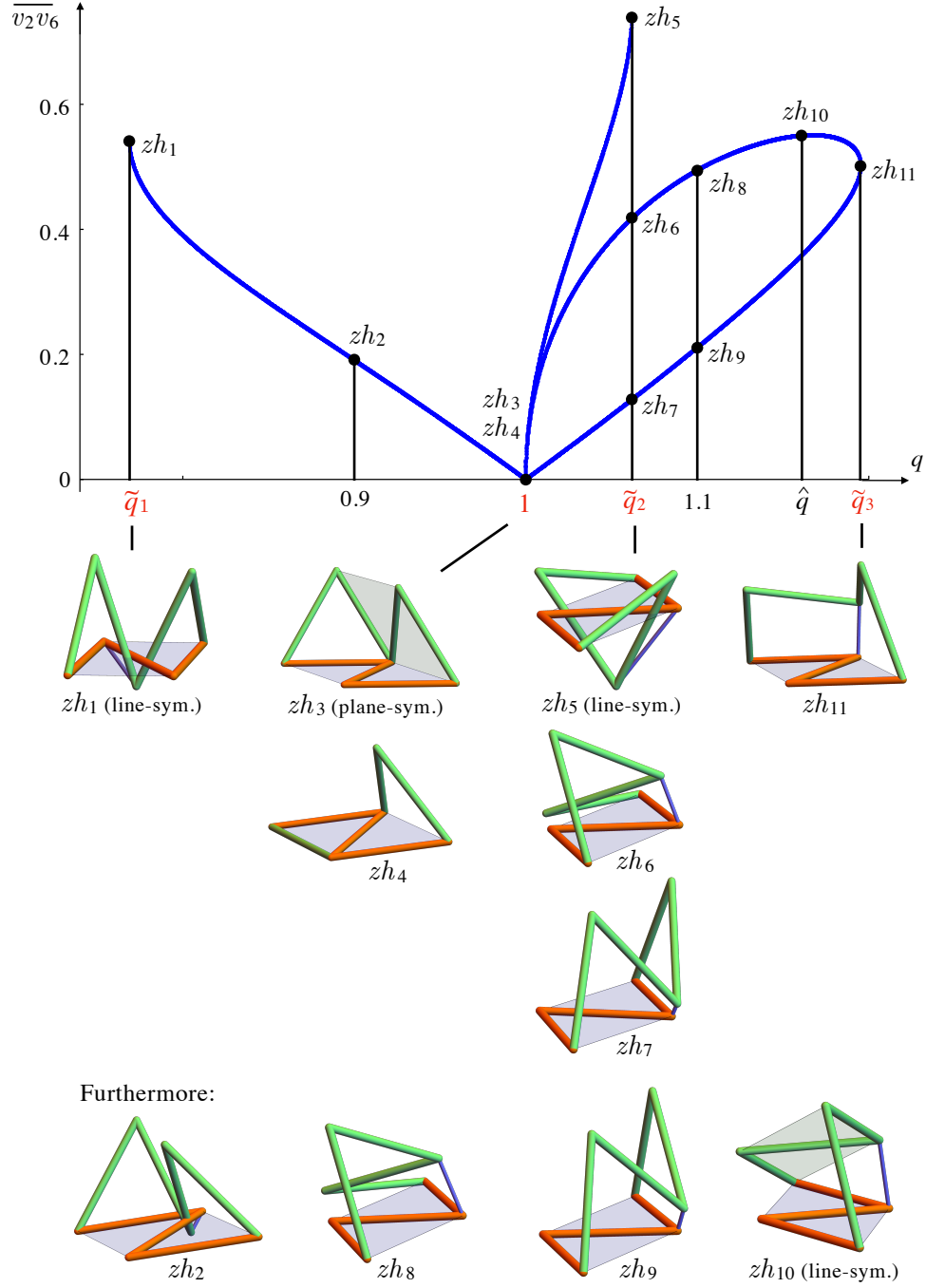


Figure 1: z-heptagons represented by diagonal pairs $(q, \overline{v_2 v_6})$ and examples.

There is a continuous transformation of z-heptagons that traverse every zh_i of Figure 1 in the order of $(zh_1, zh_2, zh_3, zh_7, zh_9, zh_{11}, zh_{10}, zh_8, zh_6, zh_4, zh_5)$. And for each $q \neq 1$, they belong to a connected component as follows:

from zh_1 until zh_4 to $C_{\text{large,up}}(q)$, from zh_4 until zh_5 to $C_{\text{small,up}}(q)$.

If the transformation is continued beyond zh_5 , as shown in Figure 2, the zh_i appear in reverse order (selected zh_i are indicated). Thus, one finally obtains the original heptagon zh_1 , so that the transformation process is closed. Note that q varies piecewise linearly.¹

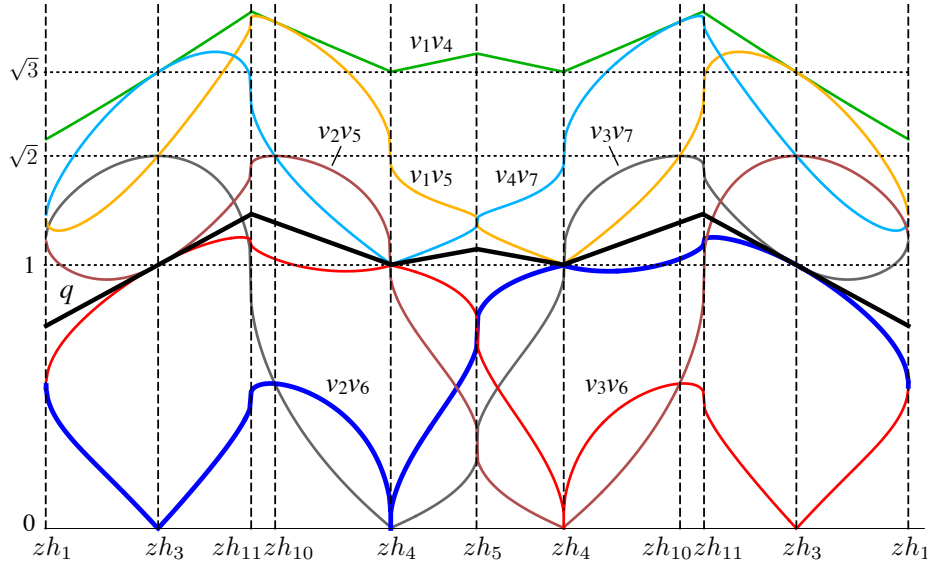


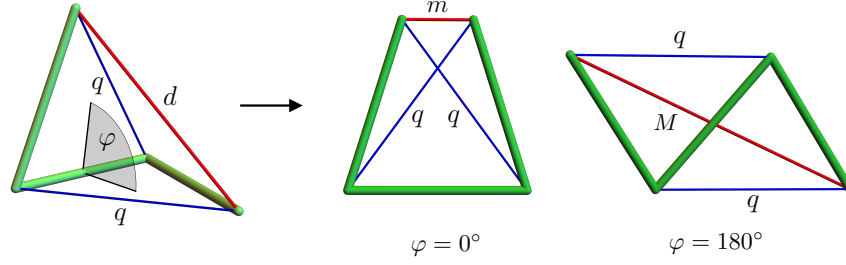
Figure 2: Continuous transformation within the z-heptagons.

Since the set of all z-heptagons contains the plane-symmetric zh_3 , it is also possible to obtain by continuous transformations the mirror-inverted z-heptagons (Lemma in [2]). Thus, we have: *The set of all z-heptagons is connected.*

5. Extreme length of main diagonals in heptagons with a fixed q

Four consecutive vertices of a heptagon form a tetrahedron, where one edge is a main diagonal whose length is denoted by d . We first consider this tetrahedron independently of the heptagon and assume that q is fixed. Then, as illustrated in the following figure, d monotonically increases from a minimum length m to a maximum M by varying φ from 0° to 180° (φ being the dihedral angle at the edge opposite to that of length d).

¹A corresponding animation can be found under the menu "Animations & Figures".



In the boundary cases we obtain an isosceles trapezoid and a parallelogram, and from the Ptolemy's theorem and the parallelogram law it immediately follows:

$$m = |q^2 - 1|, \quad M = \sqrt{2q^2 + 1}.$$

When do these boundary cases appear in heptagons? As can easily be seen, the answer to this question is as follows:

- For a fixed q , the smallest main diagonal length m is assumed in all plane-symmetric heptagons.
- For a fixed q , the largest main diagonal length M is assumed in all z -heptagons.

Plane-symmetric heptagons exist for any possible q , i.e., for q from $[q_1, q_3]$ (see [2]), whereas z -heptagons occur only for q from the subinterval $[\tilde{q}_1, \tilde{q}_3]$. Therefore, for q from $[q_1, q_3] \setminus [\tilde{q}_1, \tilde{q}_3]$, the largest main diagonal must be smaller than M ; but a characterization of these cases remains open.

References

- [1] Richardson, G.: The trigonometry of a tetrahedron, *Math. Gaz.* 2.32 (1902), 149-158.
- [2] Siegerist F., Wirth K.: Regular spatial heptagons based on symmetry, published online first in *Elem. Math.* (2023).