Angle sum of polygons in space*

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Abstract. It is examined for which angle sums a polygon in space exists.

We consider polygons in the three-dimensional Euclidean space with n generally non-coplanar vertices ($n \geq 3$) and call them *n-gons* for short. An *angle* of an n-gon is defined as the angle between adjacent sides that is smaller than or equal to $180°$. Intersecting sides, coinciding vertices, and even angles of $0°$ are permitted.

Theorem. *An n-gon in Euclidean space E^3 with angle sum S_n exists if and only if*

$$(n-2) \cdot 180° \geq S_n \geq \begin{cases} 0° \text{ for even } n \\ 180° \text{ for odd } n. \end{cases} \tag{1}$$

Proof. First, we show by induction on n that the upper bound from (1) forms a necessary condition for the existence of an n-gon. Let $S_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ be the sum of the n-gon's consecutive angles. The base case $S_3 = 180°$ is known. Adding to the n-gon a further vertex with angle α_{n+1}, as shown in Figure 1, we obtain the new vertex angles α_1' and α_n' and the triangle angles β and γ. From $(n-2)\cdot 180° \geq S_n$ and using the spherical triangle inequality, it follows by the induction step that

$$\begin{aligned} ((n+1)-2)\cdot 180° &\geq S_n + 180° = S_n + \beta + \gamma + \alpha_{n+1} \\ &= S_n - \alpha_1 + (\alpha_1 + \beta) - \alpha_n + (\alpha_n + \gamma) + \alpha_{n+1} \\ &\geq S_n - \alpha_1 + \alpha_1' - \alpha_n + \alpha_n' + \alpha_{n+1} = S_{n+1}. \end{aligned}$$

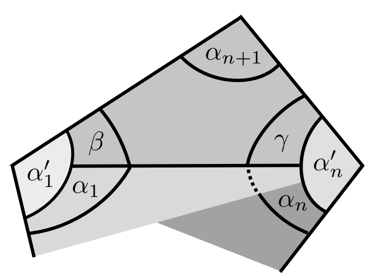

Figure 1

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As regards the necessary conditions of the lower bound from (1), it suffices to show that $S_n \geq 180°$ for odd n. To do this, we generalize an approach often used at school to prove that $S_3 = 180°$: the angles α_i of an n-gon are translated such that their vertices come to lie in a common point O and, in addition, those with even index i are reflected at O. In this way, we obtain an *angle fan* with a common side of α_i and α_{i+1} for $1 \leq i \leq n-1$, and an angle of $180°$ between the opposite sides of α_1 and α_n, as illustrated in Figure 2 for $n = 5$. Hence, again based on the spherical triangle inequality, it follows that $S_n \geq 180°$.

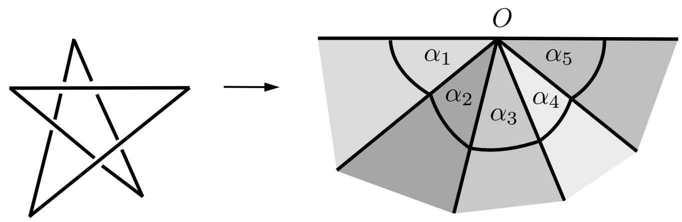

Figure 2

Next, we verify that (1) is sufficient for the existence of an n-gon by giving an example for each angle sum S_n.

For even n, consider an n-gon, as shown in Figure 3 for $n = 10$, but without point v. Its sides are diagonals of the lateral rectangles of a regular prism, and we choose their common length to be 1. This n-gon, which we call a *crown*, has equal angles. If the radius r of the circumscribed circle of the base area is continuously varied, the prism degenerates in two cases: for $r = 0$, it becomes a line segment with $S_n = 0°$, and for $r = 1/(2 \sin \frac{\pi}{n})$, it results in a regular planar n-gon and thus $S_n = (n-2) \cdot 180°$. The continuity ensures that S_n assumes all values from (1) between these boundaries.

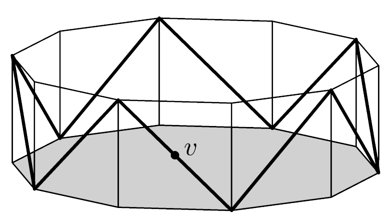

Figure 3

For odd n ($n \geq 5$), we add to a crown with $n-1$ vertices a further vertex v which is the midpoint of a side, as in Figure 3 for $n = 11$. Since the angle at v is $180°$, it follows for each r that $S_n = S_{n-1} + 180°$, and thus S_n again assumes all values from (1). $\square$

Boundaries. The upper bound $S_n = (n-2) \cdot 180°$ can only be reached if in the step of the above induction proof it holds $\alpha_1' = \alpha_1 + \beta$ and $\alpha_n' = \alpha_n + \gamma$, and consequently $\alpha_1' \leq 180°$ and $\alpha_n' \leq 180°$. The two equations imply that a corresponding n-gon is planar and the two inequalities, which in addition exclude overlapping and concavity, that it is convex.

Concerning the lower bounds, an n-gon with even n and $S_n = 0°$ is obviously linear. However, an n-gon with odd n and $S_n = 180°$ is planar, which is due to the fact that the associated angle fan must be planar. If in such an n-gon all α_i are different from $0°$, it can be characterized by having the largest turning number t, given by $t = (n-1)/2$. Figure 4 shows a heptagon with $t = 3$ and thus $S_7 = 180°$, together with the star (the great heptagram), which is the most symmetric version of the latter. An n-gon with $S_n = 180°$ and one or more vanishing angles α_i is obtained by limiting processes. If n–1 angles vanish and therefore the remaining one becomes $180°$, we get again a linear n-gon.

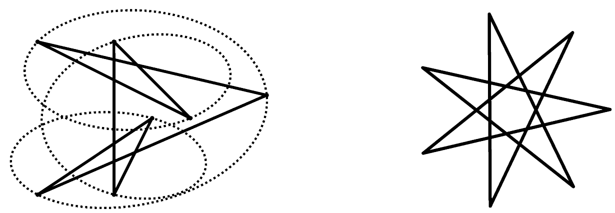

Figure 4

Summarizing the main point, we have that *an n-gon with a boundary angle sum S_n from* (1) *is planar.*

Generalization. The Theorem holds for n-gons in any Euclidean space E^d with $d \geq 2$. For $d > 3$, the proof works in the same way as in E^3. For $d = 2$, it remains to show that for each non-boundary angle sum S_n from (1) there exists a planar n-gon, which can easily be done by means of examples.

Remark. We could not find our result elsewhere in the present general form. However, for some classes of equilateral n-gons, it is implicitly contained in [1].

Reference

[1] Kamiyama Y.: A filtration of the configuration space of spatial polygons, *Adv. Appl. Discrete Math.* 22 (2019), 67–74.